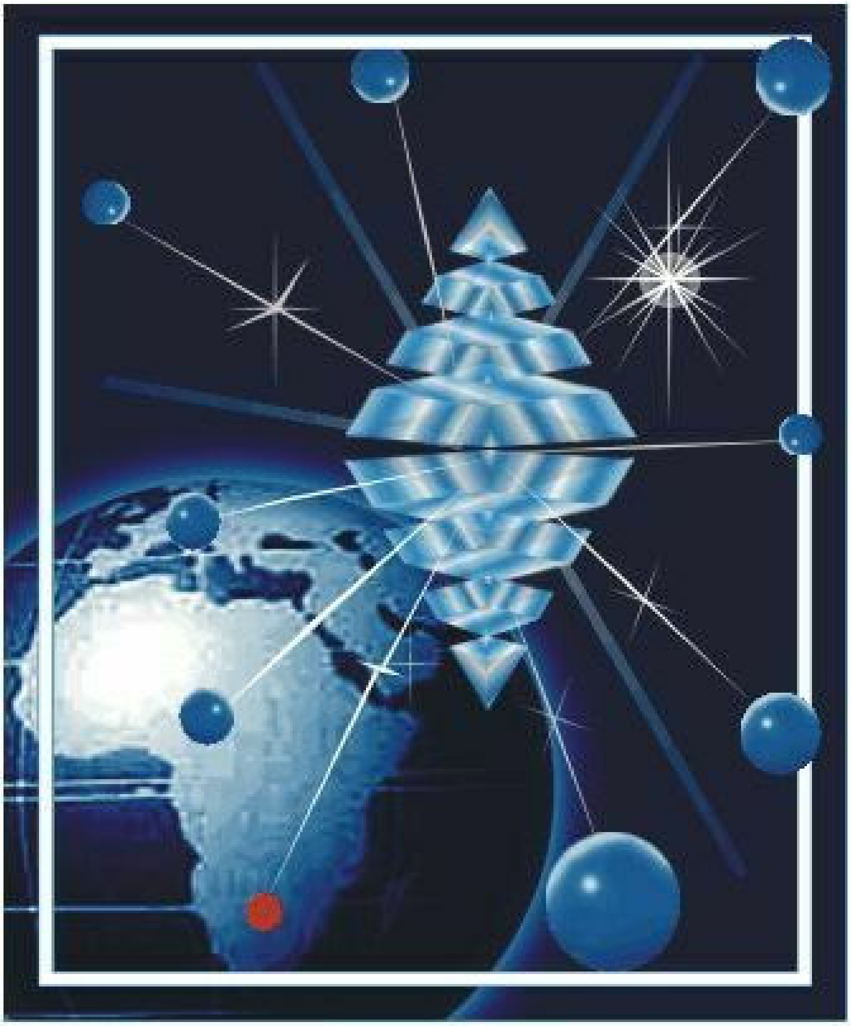

FINANCIAL CHAOS THEORY

EXOTIC OPTIONS TRADED IN THE SOUTH AFRICAN MARKETS

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Summary

This document gives a brief overview of the vanilla and exotic options that trade in the South African financial markets.

A nucleus of knowledge, experience and creativity in derivative products

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I. Introduction

Barrier options are standard calls and put except that they either disappear (the option is knocked out) or appear (the option is knocked in) if the underlying asset price breach a predetermined level (the barrier) [BB98]. Barrier options are thus conditional options, dependent on whether the barriers have been crossed within the lives of the options. These options are also part of a class of options called path-dependent options[1,2].

Apart from distinguishing between knock-in and knock-out options there is a second distinction to make. If the options knocks in or out when the underlying price ends up above the barrier level, we speak of an up-barrier. Likewise, if the price ends up below the barrier we speak of a down-barrier [CS97].

Barrier options are probably the oldest of all exotic options and have been traded sporadically in the US market since 1967 [Zh97]. These options were developed to fill certain needs of hedge fund managers. Barrier options provided hedge funds with two features they could not obtain otherwise: the first is that most "down-and-out" options were written on more volatile stocks and these options are significantly cheaper than the corresponding vanilla options. The second feature is the increased convenience during a time when the trading volume of stock options was rather low. In other words, barrier options were created to provide risk managers with cheaper means to hedge their exposures without paying for price ranges that they believe unlikely to occur. Barrier options are also used by investors to gain exposure to (or enhance returns from) future market scenarios more complex than the simple bullish or bearish expectations embodied in standard options. The features just mentioned have helped to make barrier options the most popular path-dependent options being traded world wide.

In this report we discuss *European-style barrier options* only as these can be valued analytically in a Black-Scholes environment [BS73,Sm76,Hu97,BB98] i.e., where

- where the underlying asset is assumed to follow a lognormal random walk;
- and arbitrage arguments allow us to use a risk-neutral valuation approach to obtain closed form solutions.

We also assume that the barrier is monitored continuously in time to determine whether it has been breached or not.

This report is divided as follows: in the next section we will define barrier options and look at all the different types and classes. In section 3 we discuss the process involved in monitoring the barrier levels and in section 4 we discuss the valuation of these options. In section 5 we discuss parity relationships and in section 6 we look at the payoff profiles. Section 7 is devoted to certain peculiarities of barrier options that should be considered when dealing in them. The hedging of barrier options is problematic if the spot price is close to the barrier level. The whole of section 8 is devoted to the risk parameters and hedging of barrier options. In section 9 we give reasons as to why one should use barrier options and we look at actual examples in section 10.

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[1] A path-dependent option is an option whose payoff depends on the history of the underlying asset price.

[2] Other path-dependent options are Asian options, look-back options, ladder options and chooser or shout options.

Barrier options can also be American in nature. A variety of numerical methods have been proposed to value these but this will be the topic of a forthcoming paper. Barrier options also form part of certain hedging structures in use today. Structures like the roll up put or roll down call or ladders are just made up of series of barrier options. These will be mentioned briefly but will be discussed in another paper.

II. Defining Barriers

We define two types of barriers: a barrier above the current asset price is an `*up barrier'*; if it is ever crossed it will be from below. A barrier below the current asset price is called a `*down barrier'*; if it is ever crossed it will be from above. Barrier options can also be divided into two classes: *in options* and *out options* [Ne96].

An *in barrier* (or *knock-in* option) will pay off only if the asset price finishes in-the-money and if the barrier is breached sometime before expiration. Every *knock-in* option starts inactive (it does not yet exist) and will stay inactive if the barrier is never crossed – in this situation the option expires worthless[3]. When the asset price crosses the barrier, the *in barrier* option is *knocked in* and becomes a standard vanilla option of the same type (call or put) with the payoff the same as a standard option.

An *out barrier* (or *knock-out* option) will pay off only if the asset price finishes in-the-money and the barrier is never breached before expiration – the payoff is the same as a standard option. Every *knock-out* option starts out as a standard vanilla option (call or put). Its behaviour is exactly the same as that of a vanilla option as long as the asset price never crosses the barrier. If the asset price crosses the barrier, the option is *knocked out* and it expires worthless (the option becomes *null and void* and there is no chance of recovery).

There are eight types of vanilla barrier options:

1. up-and-out call and put,
2. up-and-in call and put,
3. down-and-out call and put,
4. down-and-in call and put.

Barrier options can also have cash rebates associated with them. This is a consolation prize paid to the holder of the option when an *out barrier* is knocked out or when an *in barrier* is never knocked in. The rebate can be nothing or it could be some fraction of the premium. Rebates are usually paid immediately when an option is knocked out, however, payments can be deferred to the maturity of the option.

We talk about vanilla barrier options because variations on the basic barrier come in many types [CS97]. First, the barrier need not be active during the whole life of the option. In this case we talk of a partial barrier instead of a full barrier. A second variation concerns the monitoring frequency. It is not always necessary or desirable to check for a barrier hit continuously. Monitoring can be limited to once a day, a

[3] This means an investor buys an option that is worthless. This option will only be of any value when the barrier is crossed and the payoff is then the same as that of an ordinary vanilla option.

week or month. In that case we speak of a discrete barrier and not a continuos barrier. Thirdly, the barrier might not necessarily be linked to the underlying price. It may be linked to another variable like another interest rate or another exchange rate. This is referred to an outside barrier as opposed to an inside barrier [Zh95]. Fourthly, we can let the barrier change over time. These are called moving barrier options and Kunitomo and Ikeda derived closed form solutions for linear barriers [KI92] while Rogers and Zane handled generalised this [RZ99]. These are called time-dependent barrier options. All of these exotic types of barrier options are not discussed here.

III. Monitoring the Barrier

In practise one has to define precisely what it means for the barrier of an option to be crossed. The issue is how the spot price of the underlying is tracked. Is the barrier breached the moment the spot price crosses it intra-day? Further, does one use the *last trade*, the *bid*, the *offer* or the *middle of the double*? One can also use the official end-of-day closing prices meaning the barrier is only deemed breached if the closing price crossed the barrier. One can also specify that the price of the underlying should have breached the barrier level by at least a certain time period. These options are called Parisian options [HS97,Wi98].

The key to barrier event monitoring is transparency [Hs97]. The option writers need to be transparent as to what method is used to monitor whether a barrier has been breached or not. This process needs to be impartial, objective and consistent. For instance, when the *last trade* is used as monitor, the minimum size of the transaction needed to trigger a barrier event becomes crucial. This is to prevent dealers from trying to push the spot price through the barrier level, at their own benefit.

Option writing warehouses need to put policies in place to prevent dealers from deliberately triggering or defending barriers.

IV. Pricing

Merton was first at deriving a closed-form solution for a barrier option where he showed that a European barrier option can be valued in a Black-Scholes environment [Me73]. Thereafter Rubinstein and Reiner generalised barrier option-pricing theory [RR91]. Rich gave an excellent summary of barrier options [Ri94]. With a rebate continuous dividend yield and continuous monitoring of the barrier[4], the following equations are obtained:

[4] Broadie and Glasserman [BG 97] gives a simple modification to adjust the prices if the barrier is monitored discretely in time e.g., daily or weekly.

$$
\left.
\begin{aligned}
A &= \phi S\, e^{-d\tau}\left(\frac{H}{S}\right)^{2\lambda} N(\eta y) - \phi K\, e^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y - \eta\sigma\sqrt{\tau}\right)\\
B &= R\, e^{-r\tau}\left[N\left(\eta x_1 - \eta\sigma\sqrt{\tau}\right) - \left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y_1 - \eta\sigma\sqrt{\tau}\right)\right]\\
C &= \phi S\, e^{-d\tau} N(\phi x) - \phi K\, e^{-r\tau} N\left(\phi x - \phi\sigma\sqrt{\iota}\right)\\
D &= \phi S\, e^{-d\tau} N(\phi x_1) - \phi K\, e^{-r\tau} N\left(\phi x_1 - \phi\sigma\sqrt{\tau}\right)\\
E &= \phi S\, e^{-d\tau}\left(\frac{H}{S}\right)^{2\lambda} N(\eta y_1) - \phi K\, e^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y_1 - \eta\sigma\sqrt{\tau}\right)\\
F &= R\left[\left(\frac{H}{S}\right)^{a+b} N(\eta z) + \left(\frac{H}{S}\right)^{a-b} N\left(\eta z - 2\eta b\sigma\sqrt{\tau}\right)\right]
\end{aligned}
\right\} \qquad (1)
$$

where S is the spot market price, K is the strike price, H is the barrier (in the same units as S and K), R is the rebate (in currency units), τ is the annualised time till expiration, r is the risk-free short term interest rate in continuous format, d is the dividend yield in continuous format, σ is the volatility and ϕ and η are binary variables set out in Table 1.
All the other variables are defined as follows (with $\ln$ the natural logarithm)

$$
\left.
\begin{aligned}
x &= \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{S}{K}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\}\\
x_1 &= \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{S}{H}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\}\\
y &= \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H^2}{SK}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\}\\
y_1 &= \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H}{S}\right) + \left(r - d + \frac{\sigma^2}{2}\right)\tau\right\}\\
z &= \frac{1}{\sigma\sqrt{\tau}}\left\{\ln\left(\frac{H}{S}\right) + b\sigma^2\tau\right\}\\
\lambda &= 1 + \frac{\mu}{\sigma^2}\\
a &= \frac{\mu}{\sigma^2}\\
b &= \frac{1}{\sigma^2}\left[\sqrt{\mu^2 + 2r\sigma^2}\right]\\
\mu &= r - d - \frac{\sigma^2}{2}.
\end{aligned}
\right\} \qquad (2)
$$

$N(\bullet)$ is the cumulative of the normal distribution function given by [Hu97,BB98]

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{z^2}{2}\right) dz \,.$$

An excellent numerical routine to value $N(x)$ was given by Cody [Co96]. He uses Chebyshev approximations.

The valuation formulas for the eight barrier options can be written as combinations of the quantities A to F given in Equation (1). The value of each barrier is also dependent on whether the barrier H is above or below the strike price K.

Call	Put
Down and In Barriers	
$\phi = \eta = 1$ $DIC_{K \geq H} = A + B$ $DIC_{K<H} = C - D + E + B$	$\phi = -1, \eta = 1$ $DIP_{K \geq H} = D - A + E + B$ $DIP_{K<H} = C + B$
Up and In Barriers	
$\phi = 1, \eta = -1$ $UIC_{K \geq H} = C + B$ $UIC_{K<H} = D - A + E + B$	$\phi = -1, \eta = -1$ $UIP_{K \geq H} = C - D + E + B$ $UIP_{K<H} = A + B$
Down and Out Barriers	
$\phi = \eta = 1$ $DOC_{K \geq H} = C - A + F$ $DOC_{K<H} = D - E + F$	$\phi = -1, \eta = 1$ $DOP_{K \geq H} = C - D + A - E + F$ $DOP_{K<H} = F$
Up and Out Barriers	
$\phi = 1, \eta = -1$ $UOC_{K \geq H} = F$ $UOC_{K<H} = C - D + A - E + F$	$\phi = -1, \eta = -1$ $UOP_{K \geq H} = D - E + F$ $UOP_{K<H} = C - A + F$

Table 1: Pricing Formulas for European barrier options. The variables are defined in Equation (1)

The abbreviations used are: $DIC_{K<H}$ is short for "down and in call barrier option" where the strike value K is less than the barrier value H. If the payment of the rebate is deferred to maturity for the *knock-out* options, we put $F = B$ in the equations above.

The following should be considered when implementing the formulas in Table 1:

- The formulas given for the "in"-barriers are used to value the option before the barrier is hit. After the barrier has been hit (the option was knocked in), the buyer has an ordinary vanilla call or put. Use the Black-Scholes equations for vanilla options to value the option.
- The *knock-out's* value is zero after the barrier has been hit.

V. Parity Relationship

In-out parity is the barrier option's answer to put-call parity: if we combine an *in- and an out- barrier option,* both with the same strike and expiration date, we get the price of a vanilla option [Ch97]. Here is how it works.

Consider a portfolio consisting of an *in-option* and its corresponding *out-option* without any rebate. If the barrier is never reached, the in-option will end up worthless, yet the out-option will have its corresponding vanilla option as payoff. On the other hand, if the barrier is reached any time during the life of the option, the in-option will have the corresponding vanilla option as its payoff whereas the out-option will end up worthless. Since the portfolio and the corresponding vanilla option have exactly the same payoff at maturity, arbitrage arguments guarantee that the portfolio and the vanilla option must be the same. Algebraically this identity is

Black-Scholes vanilla Call (Put) = Out Call (Put) + In Call (Put)

VI. Premium Profiles

Figures 1&2 show the premium profiles of two types of barrier options. We have the following: All Share Top 40 Index call/put, $S = 6800,\quad K = 6800,\quad R = 0,\quad \sigma = 28\%$, $r = 15\%,\quad d = 2\%,\quad t = 1$.

Figure 1: Down and Out Put, $H = 5500$

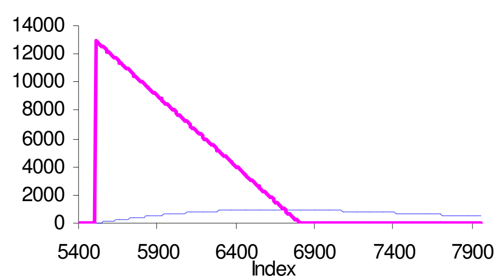
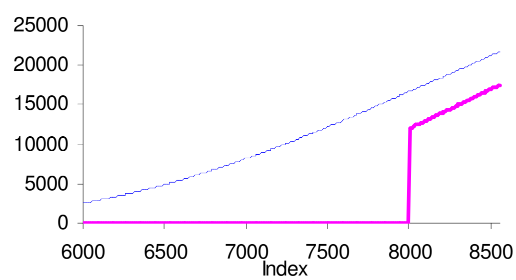


Figure 2: Up and In Call, $H = 8000$



Current value of barrier option

Value at expiry of barrier option

From the profiles it is apparent that the expiration values can change abruptly. This makes the hedging very difficult.

VII. Behaviour of Barrier Options

Barrier options are less expensive than standard options but provide similar potential investment returns. We list some points that need to be considered when trading barrier options:

Knock-outs:

- The closer the barrier level to the current spot, the lower the barrier option's premium;
- The higher the volatility, the lower the barrier option's premium;
- The longer the time to expiration of the option, the lower the barrier option's premium.

For example, an investor that is long an up-and-out call, is forfeiting some of the upside potential of an ordinary call but the payoff can be the same if the barrier is never hit. If the barrier is hit, the investor looses his exposure and the barrier must thus be less expensive than a standard option.

Knock-in options' behaviour is similar to standard options but the premiums are also less:

- The closer the barrier level to the current spot, the higher the barrier option's premium;
- The higher the volatility, the higher the barrier option's premium;
- The longer the time to expiration of the option, the higher the barrier option's premium.

VIII. Hedging Barrier Options

The risk parameters can be obtained by calculating the relevant partial derivatives of Equation (1). To obtain the deltas (Δ) we calculate the partial derivatives of the functions A to F with respect to S and by substituting these in the equations given in Table 1 e.g. substitute A with $\partial A/\partial S$ give Equation (3). By taking the partial derivatives, with respect to S, we obtain:

$$
\left.
\begin{aligned}
\frac{\partial A}{\partial S} &= \frac{2}{S}(1-\lambda)A - \phi\ e^{-d\tau}\left(\frac{H}{S}\right)^{2\lambda} N(\eta y) \\
\frac{\partial B}{\partial S} &= \frac{2}{S}(1-\lambda)R\ e^{-r\tau}\left(\frac{H}{S}\right)^{2\lambda-2} N\left(\eta y_1 - \eta\sigma\sqrt{\tau}\right) \\
&\quad + \frac{\eta R}{\sigma\sqrt{\tau}H}\ e^{-d\tau}\left[N'(\eta x_1) + \left(\frac{H}{S}\right)^{2\lambda} N'(\eta y_1)\right] \\
\frac{\partial C}{\partial S} &= \phi\ e^{-d\tau} N(\phi x) \\
\frac{\partial D}{\partial S} &= \phi\ e^{-d\tau} N(\phi x_1) - \phi K\ e^{-d\tau}\frac{N'(\phi x_1)}{\sigma\sqrt{\tau}}\left[1 - \frac{K}{H}\right] \\
\frac{\partial E}{\partial S} &= \frac{2}{S}(1-\lambda)E - \phi\ e^{-d\tau}\left(\frac{H}{S}\right)^{2\lambda} N(\eta y_1) - \phi\eta\ e^{-d\tau}\left(\frac{H}{S}\right)^{2\lambda}\frac{N'(\eta y_1)}{\sigma\sqrt{\tau}}\left[\frac{K}{H} - 1\right] \\
\frac{\partial F}{\partial S} &= -\frac{R}{S}\left\{(a+b)\left(\frac{H}{S}\right)^{a+b} N(\eta z) + (a-b)\left(\frac{H}{S}\right)^{a-b} N\left(\eta z - 2\eta b\sigma\sqrt{\tau}\right)\right. \\
&\quad \left. + \frac{\eta}{\sigma\sqrt{\tau}}\left[\left(\frac{H}{S}\right)^{a+b} N'(\eta z) + \left(\frac{H}{S}\right)^{a-b} N'\left(\eta z - 2\eta b\sigma\sqrt{\tau}\right)\right]\right\}
\end{aligned}
\right\}
\qquad (3)
$$

Here, $N'(\bullet)$ is the cumulative normal probability function defined as

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The gammas (Γ) are obtained by taking the partial derivatives of the equations in Equation (3) with respect to S i.e., one calculates the partial derivatives of the Δ's or the second partial derivatives of the quantities in Equation (1) e.g. $\partial^2 A / \partial S^2$.

Using delta and gamma to hedge barrier options can be difficult and dangerous. The delta becomes very large and the gamma becomes infinite if we are very close to the barrier.
The delta and gamma also change abruptly when the option is knocked in or out. This is illustrated in Figures 3 and 4.

Figure 3: Delta of barrier in Figure 1

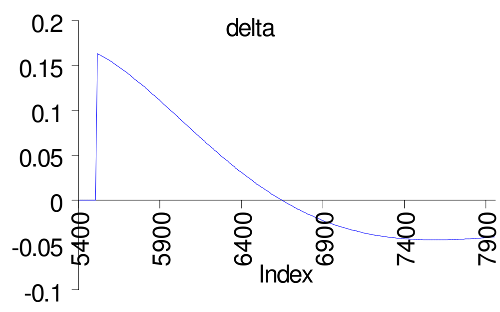
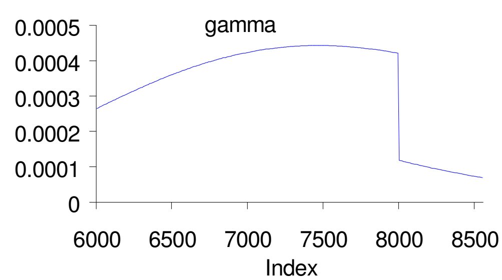


Figure 4: Gamma of barrier in Figure 2



Some barriers are very difficult to hedge statically[5]. With *put-call-symmetry* [BC94,Wi98], however, we are able to create perfect static replications for some barriers (if $R=0$) with zero cost strategies – this works best where the underlying instrument is a futures contract (put $r=d$ in Equation. 1 [Hu97]). These hedging techniques are easily implemented and are described in the Table 2 where we have assumed we are the writers of the barrier options and we want to hedge our exposure.

Up and In Call ($K \geq H$) Do nothing until barrier is hit. The moment the barrier is hit, buy a vanilla call with the same strike as that of the up and in call originally sold.
Down and In Put ($K \leq H$) Do nothing until the barrier is hit. The moment the barrier is hit buy a put with the same strike as that of the down and in put originally sold.
Down and In Call ($K \geq H$) With the sale of the barrier, go long K/H puts with strike H^2/K. The moment the barrier is touched, sell the puts and buy a call at the same strike as that of the down and in call. If the barrier is never touched, all options expire worthless.
Down and Out Call ($K \geq H$) With the sale of the barrier, go short K/H puts with strike H^2/K and buy a call with strike K. The moment the barrier is touched, sell the calls and buy back the puts.
Up and In Put ($K \leq H$) With the sale of the barrier, go long K/H calls with strike H^2/K. The moment the barrier is touched, sell the calls and buy a put at the same strike as that of the up and in put. If the barrier is never touched, all options expire worthless.
Up and Out Put ($K \leq H$) With the sale of the barrier, go short K/H calls with strike H^2/K and buy a put with strike K. The moment the barrier is touched, sell the puts and buy back the calls at the same strike as that of the up and out put.

Table 2: Hedging strategies for barrier options.

The strategies in Table 2 are zero cost if everything stays the same i.e., interest rates and volatilities. This hardly ever happen which means there are some risks involved in doing this. In illiquid or underdeveloped markets, one might also not be able to pick up the vanilla options at the strikes proposed. For instance, say that some Alsi future is currently trading at 8 000 and we want to hedge an up-and-in put with strike 8 000 and barrier level of 11 000. This means we must buy calls with a strike at 12 763. This might not be possible or due to the volatility skew it might be too expensive.

Statically hedging barrier options not mentioned in Table 2 is impossible; those can only be hedged dynamically i.e., re-value and manage risk on a daily basis. Another hedging strategy often used is, if the number of barrier options is small, all can be

[5] To hedge statically means to put a hedge in place and then forget about it till expiry.

dumped into a large portfolio of standard vanilla options and the risk is then managed all together.

Delta hedging becomes very difficult if the time to expiry is short and the spot price is near the barrier level [Sh96]. For a knock out, the delta can become very negative near the knock out boundary. The hedger is in an unstable situation. Because the delta is so negative he should take a very large short position in the underlying stock and invest these proceeds in the money market. If the stock does not cross the barrier he covers his short position with the money market funds , pays off the option and is left with zero funds – the option would be alive and would expire in-the-money. If the stock move across the barrier (the option is knocked out) the delta becomes zero. He should now cover his short position with the money market. This is more expensive than before because the stock price has risen and consequently he is left with no money. But, the option is not alive anymore so no money is needed to pay it off.

Because a large short position is being taken, a small error in hedging can create a significant effect. To circumvent this, do the following: rather than using the boundary condition $v(t,H)=0,\ 0\le t\le T$, (i.e. the value of the barrier option should be zero at the barrier level), solve the barrier partial differential equation with the new boundary condition

$$v(t,H)+\alpha H\Delta(t,H)=0, \qquad 0\le t\le T$$

where α is a "tolerance parameter", say 1%. At the boundary, $H\Delta(t,H)$ is the Rand value of the short position. The new boundary condition guarantees:

- $H\Delta(t,H)$ remains bounded;
- the value of the portfolio is always sufficient to cover a hedging error of α times the Rand value of the short position.

IX. The uses of Barriers

Barrier options provide an array of opportunities to create highly structured risk-reward or pay-off profiles consistent with asset-price expectations as well as asset-price path patterns [Da96]. There are five basic reasons to use barrier options rather than standard options [DK93]:

A. The premiums of barrier options are generally lower than that of standard options.

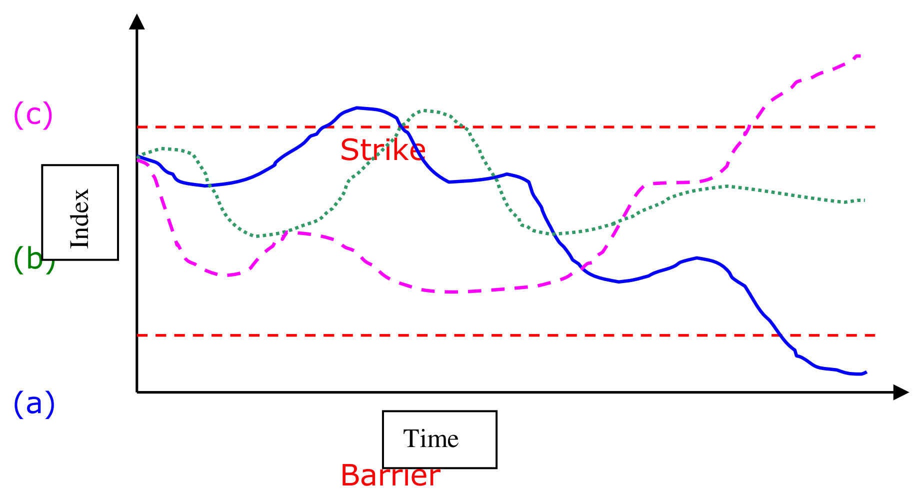
The premium of a barrier option is a function of the barrier level. The premium can thus be manipulated by shifting the barrier to where the client wants it to be. If you think the chance of knockout is small you can take advantage of the lower premium and get the same benefits. The same holds for knock-in options if you feel the chance that the barrier will be breached is very high. This explains much of the appeal of barrier options: they can be tailored to fit an investor's view of the market. This is depicted in Example 1.

B. Barrier options may have payoffs that more closely match your beliefs about the future behaviour of the market.

By buying a barrier option, the investor can eliminate paying for those market scenarios he/she thinks are unlikely to occur. Alternatively, an investor can enhance a portfolio's return by selling a barrier that pays off only on scenarios he/she thinks are improbable - this is illuminated through Example 2

Example: Suppose current All Share Top 40 index is at 7000 and you believe that the market is going to strengthen pushing the index higher to 8000 over the next year. However, you also believe that if the index falls to 6500, it will fall further. If one buys a *down and out call*, struck at 7250 that is knocked out, at any time, if the market falls to 6500, one avoids paying for scenarios in which the index decreases but one still benefits from scenarios in which the index increases - at a reduced premium. For the paths in Figure 5, path (a) leads to knock-out, path (b) results in option expiring out-the-money and path (c) results in option expiring in-the-money.

Figure 5: Possible paths of the index over the next year



C. Barrier options may match your hedging needs more closely than similar standard options.

A portfolio can be hedged against downward market movements by buying knock-in options - this is illuminated through Example 3 in the following section.

D. Replicating spreads and saving transaction costs.

Consider the case of a modestly bullish investor who wants to use options to increase upside exposure, but at a low premium cost. Using standard options, the investor might put on a bullish vertical spread. This spread has two unattractive properties: first, the investor must transact in two different options, increasing the transaction cost; second, the investor knows ahead of time that the strategy is to unwind both legs simultaneously, but it may be difficult, or at least expensive, to do this in practice. On the other hand, a *down and out call*, with a suitable rebate R would also provide upside exposure, with the transaction cost of only a single option, and with a no cost "unwinding" if the barrier is hit.

Writing the down and out call has one clear advantage over taking the reverse side of the bullish vertical spread --- namely, the writer is not exposed to unlimited liability. For investors with a bearish view, this position could be attractive, since they can specify the size of their loss in case they are wrong.

E. Market Reversals

Down-and-in puts and *up-and-in calls* can be attractive substitutes to standard options for investors expecting a movement in one direction followed by a strong reversal.

X. Examples

To show the differences in premiums between barriers and standard options, we look at the following examples:

Example 1.

On 20 January 1998 the market is starting to move up. The spot market is at 5069. A portfolio manager needs protection for this upward move (for the next two months) but if the market goes beyond 5500, the portfolio does not need the protection anymore. At that date the March 1998 Alsi future was trading at 5209. This manager can either buy a vanilla call or a *up-and-out call* . The next table lists the differences in premiums:

Vanilla OTC Alsi futures Call		European up-and-out Alsi futures call	
S:	5209	S:	5209
K:	5209	K:	5209
σ:	28%	H:	5500
r:	15%	σ:	28%
Expiry Date:	20 March 1998	r:	15%
Deal Date	20 January 1998	Expiry Date:	20 March 1998
		Deal Date	20 January 1998
Premium:	R2 285.92	Premium:	R35.67

On 20 March 1998, the Alsi futures contract closed out at 6186 and this manager protected his/her portfolio up to 5500 at a saving of 98%.

Example 2.
On 20 January 1998 a portfolio manager want exposure to the market because the market is moving up. At that date the March 1998 Alsi future was trading at 5209. He/she thinks this is the start of a bull run that will carry the market beyond the 6000 mark. This manager can either buy a vanilla call or a *up-and-in call* . The next table lists the differences in premiums:

Vanilla OTC Alsi futures Call		European up-and-in Alsi futures call	
S:	5209	S:	5209
K:	5209	K:	5209
σ:	28%	H:	6000
r:	15%	σ:	28%
Expiry Date:	20 March 1998	r:	15%
Deal Date	20 January 1998	Expiry Date:	20 March 1998
		Deal Date	20 January 1998
Premium:	R2 285.92	Premium:	R1 516.92

On 20 March 1998, the Alsi futures contract closed out at 6186 and this portfolio was exposed to the market at a saving of 33%.

Example 3.
On 20 June 1997 spot market closed at 6302 and the closing price for the December Alsi futures contract was 6940. A portfolio manager, who is long, is becoming nervous and he/she wants protection in case the market crashes to below 5500. This manager can either buy a vanilla put or a *down-and-in put*. The next table lists the differences in premiums:

Vanilla OTC December Alsi put		OTC down-and-in Alsi put	
S:	6940	S:	6940
K:	6940	K:	6940
σ:	24%	H:	5500
r:	15%	σ:	24%
Expiry Date:	19 December 1997	r:	15%
Deal Date	20 June 1997	Expiry Date:	19 December 1997
		Deal Date	20 June 1997
Premium:	R4 371.04	Premium:	R2 568.91

On December 19 1997, the December Alsi futures contract closed out at 5163 and this portfolio was protected at a saving of 68%.

XI. Structures Using Barriers

A. Roll Up Puts and Roll Down Calls

These structures appeal to investors who feel that they may be early in implementing a bullish or bearish position i.e., they may feel the market will go down or up with sudden reversals later on. They are combinations of barrier down-and-out calls and up-and-out puts. Roll-ups and -downs are synthetics hedging structures that are put in place on top of existing long or short positions in the underlying – these are overlay structures.

The value of the roll down call can be calculated as follows: let $H_1, H_2, \ldots, H_n$ be a decreasing sequence of positive barrier levels. Similarly, let $K_0, K_1, \ldots, K_n$ be a decreasing sequence of strikes, with $K_i \geq H_i, i = 1,2,\ldots n$. The roll down call is decomposed into down-and-out calls such that[2]

$$RDC(K_i, H_i) = DOC(K_0, H_1) + \sum_{i=1}^{n} \left[DOC(K_i, H_{i+1}) - DOC(K_i, H_i) \right]$$

where $DOC(K_i, H_i)$ is a down-and-out call with strike K_i and barrier level at H_i. For a roll up put we have an increasing set of barrier levels where $K_i \leq H_i, i = 1,2,\ldots n$ such that

$$RUP(K_i, H_i) = UOP(K_0, H_1) + \sum_{i=1}^{n} \left[UOP(K_i, H_{i+1}) - UOP(K_i, H_i) \right].$$

These structures are depicted in Figures 5 and 6.

Figure 5: Roll Down Call

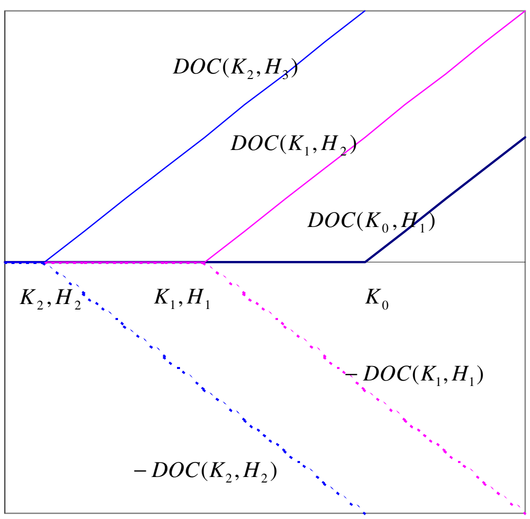
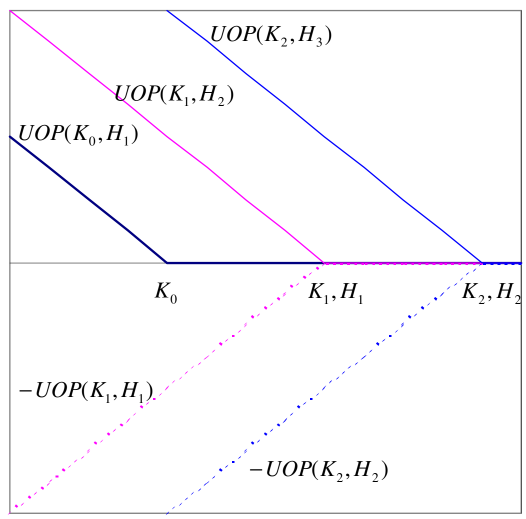


Figure 6: Roll Up Put



To explain the mechanics of a roll-up put lets look at an example where there are two roll-up points. We thus have the current market level at K_0 and two levels H_1 and H_2 where $H_1, H_2 > K_0$. The value of the roll-up put given by

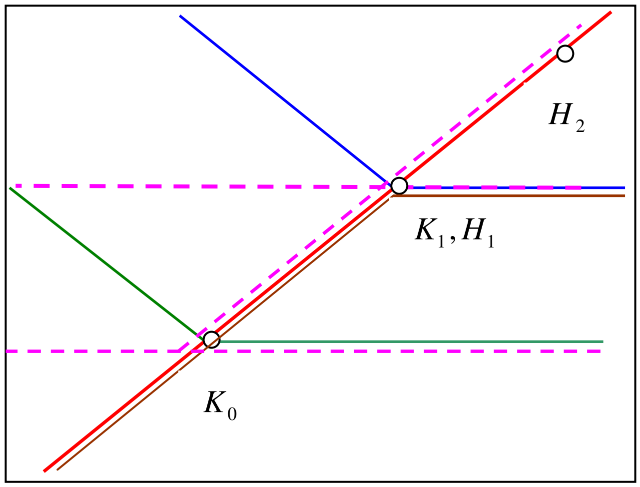
$$RUP(K_0) = UOP(K_0, H_1) + UOP(K_1, H_2) - UOP(K_1, H_1)$$

where $K_1 = H_1$.

The buyer of this roll-up put option buys an up-and-out-put struck at K_0 which vanishes as soon as the barrier at H_1 is breached. This put is active and behaves just like a vanilla put which means the investor's long position in the underlying is immediately hedged against the market turning down. He/she also buys an up-and-out-put struck at $K_1 = H_1$ which vanishes as soon the barrier at H_2 is breached.

He/she then shorts an up-and-out-put struck at K_1 that will vanish once the level H_1 has been breached. Graphically this is be depicted in Figure 9. The thick line is the original long position and the thin lines depict the up-and-out-puts.

Figure 9: a roll-up put structure



The dashed line in Figure 9 shows the possible payoffs that are possible. If H_1 is never breached this structure behaves like an ordinary vanilla call (the put struck at K_0 together with the long position in the underlying gives a synthetic call struck at K_0). This means that the investor is immediately hedged against the market moving below K_0. Also, the long and short puts struck at K_1 cancel one another. The payoff is thus given by

$$S_T + \max(0, K_0 - S_T)$$

where S_T is the value of the underlying at expiry.

If H_1 is breached the following happens: the long put struck at K_0 and the short put struck at K_1 are knocked out. In essence the strike of the synthetic call has thus moved up to K_1 and the investor is hedged such that if the market moves below K_1, he/she will at least receive K_1 - K_1 is thus locked in/guaranteed to the holder of the underlying long position and the payoff is given by

$$S_T + \max(0, K_1 - S_T)\,.$$

If H_2 is breached the whole synthetic hedging structure vanishes and the investor is left with the original long position with payoff just S_T. To circumvent this from happening it is advised that the last roll-up level be placed so far from the original level at K_0 that the chances of it being breached is nearly zero. On the other hand, the investor might decide that once the last level is breached, the original long position is so far in-the-money that he/she might risk market conditions and a subsequent reversal.

The following 3 legged example also explains the mechanics of a roll-up put: a fund manager is long the ALSI and sees that the market is going to roar up during the next 3 years. He/she also expects some corrections to take place though. In order to hedge his/her portfolio for this period of time, he/she does the following:
Say the current market is at 7000, buy a roll up put with $H_1 = 9000$, $H_2 = 11000$ and $H_3 = 13000$ where $K_0 = 7000$ and $K_1 = H_1, K_2 = H_2$. Thus, he/she basically has one put i.e., $UOP(K_0, H_1)$; the other long positions are cancelled by the short positions. If the market moves up to 9000, the long $UOP(K_0, H_1)$ and the short $UOP(K_1, H_1)$ are knocked out (cancelled) leaving the $UOP(K_1, H_2)$ active. The other long and short positions cancel one another still. He/she has thus moved his/her hedge position up to the at-the-money level without incurring extra trading costs. The same holds if the market moves beyond the 11000 level. His/her position is only cancelled once the market gets to the 13000 level.

This strategy saves quite a bit of money compared to buying an at-the-money put today and by rolling the position over to a new at-the-money put if the market gets to 9000 (and by doing the same if the market hits 11 000). It also ensures that if the market hits one of the barrier levels, and then suddenly retracts, that the portfolio is fully hedged.

As a real example let's look at a 5 year SAPPI roll-up put. SAPPI is currently trading at R52.80, the current volatility is 45%, interest is 15% and the dividend yield is 1.887%.
We assume the investor is long the SAPPI stock and wants to lock in the gains after the stock has gone up by certain amounts. The following lists a matrix of prices where the lock-in levels are 25%, 50%, 75% and 100% above the current price.

	Rungs	**Rungs**	**Rungs**	**Rungs**	**Vanilla ATM put**
	66.00	66.00	66.00	66.00	
	1,000.00	79.20	79.20	79.20	
		1,000.00	92.40	92.40	
			1,000.00	105.60	
				1,000.00	
Value	**7.92**	**9.89**	**11.60**	**13.08**	**5.62**
Value as %	14.9958%	18.7380%	21.9713%	24.7788%	10.646%

Here, the price is given for an increasing number of levels and the last level is so far out that the probability that it is ever reached is virtually zero. In the last column we have given the value of an ATM vanilla put. The roll-ups are more expensive because of the lock-in feature at every rung/level.

B. Ladder/Ratchet Option

There has recently been a lot of interest in OTC equity derivatives which guarantee the return of capital invested or allow the purchaser to periodically lock in gains. One way to achieve this is by using ladder options. Ladder options are similar to a lookback options. The buyer of a fixed strike lookback call buys an option at a certain strike, say K_0. Now, as soon as a more advantageous level is registered, the strike is adjusted to this level, say K_1 where $K_0 < K_1$. This means that the difference between the new strike and original strike level, ($K_1 - K_0$), is locked in and guaranteed to the holder of the lookback. It is, however, also possible to do this only if the difference with the original strike exceeds a certain minimum predetermined value. This is then a ladder option.

A ladder option has a set of N predetermined levels (rungs) $L_i, i = 1,2,\ldots,N$ where, every time the market breaches a certain ladder level, the intrinsic value at that level is locked in and will be paid out to the holder at expiry. A European ladder will have the following payoff[1]

$$LO = \max[\phi(S_T - K), \max\{\phi(L_i - K), 0\}, 0]$$

where LO is the terminal value of the ladder option, S_T is the value of the underlying quantity at expiry time T, K is the initial strike price, $\phi = 1$ for a call and $\phi = -1$ for a put and L_i is the i^{th} ladder level reached in the life of a option. This function shows that the pay-off for a ladder option is the greater of a plain vanilla call option with strike K and the highest ladder level reached, or zero.

As an example, look at Figure 7. This shows paths (a), (b), (c) and (d) that are different possibilities of the underlying over time. L_1 and L_2 are two rungs of a two-rung ladder call option. The payoff for path (a) at expiry is the terminal spot value minus the strike ($S_T - K$ - the same as for a vanilla call); the payoff for path (b) at expiry will be $L_2 - K$; the payoff for path (c) at expiry will be $L_1 - K$; and the payoff for path (d) is zero. The payoff is thus similar to that of a lookback option with discrete lock-in levels. In the limit when the number of rungs, $N \to \infty$ we get the vanilla fixed strike lookback option.

A ladder is a total synthetic instrument that gives synthetic exposure to the market. The exposure is obtained through the vanilla call and the gains are locked in by using barrier options.

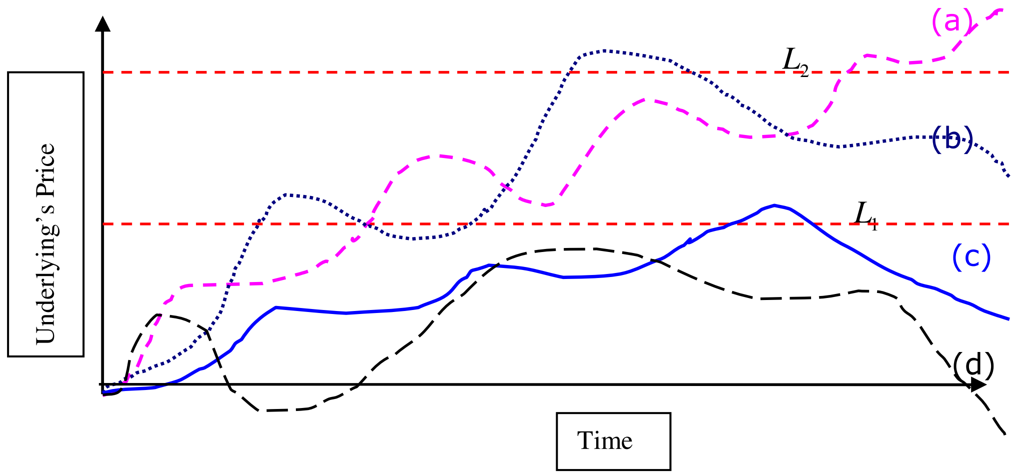
To synthesise a ladder call with initial strike K_0, we let $L_0, L_1, L_2, \ldots, L_N$ be a predetermined increasing set of barrier levels such that $K_0 = L_0$, then[2]

$$LC(K_0, L_i) = C(K_0) + \sum_{i=1}^{N} [UIP(L_i, L_i) - UIP(L_{i-1}, L_i)];$$

and for a ladder put we have a predetermined decreasing set of barrier levels $L_0, L_1, L_2, \ldots, L_N$ where $K_0 = L_0$ and thus

$$LP(K_0, L_i) = P(K_0) + \sum_{i=1}^{N} [DIC(L_i, L_i) - DIC(L_{i-1}, L_i)]$$

Figure 7: Possible price paths of underlying



where $C(K_0)$ and $P(K_0)$ is a vanilla call and vanilla put struck at K_0 respectively and $UIP(x, y)$ is an up-and-in-put with strike x and barrier level y and $DIC(x, y)$ is a down-and-in-call with strike x and barrier level y.

This shows that ladder options are packaged from ordinary vanilla and barrier options. A ladder option is cheaper than a lookback but more expensive than the corresponding vanilla option. The reason for this is that a ladder consists of a vanilla option and two barrier options. Ladders are hedged by using the strategies given for barrier options.

To explain the mechanics of a ladder lets look at an example where there is only one rung at level L_1 such that $L_1 > K$ and $L_0 = K$. The value of this ladder call is given by

$$LC(K \ , L_1) = C(K \ ) + UIP(L_1, L_1) - UIP(L_0, L_1) .$$

The buyer of a ladder call option buys a vanilla call, struck at K; he/she also buys an up-and-in-put, struck at L_1, that will only be activated once the underlying has breached the level at L_1. He/she then shorts an up-and-in-put struck at K that will be activated once the level L_1 has been breached. Graphically this is be depicted in Figure 8.

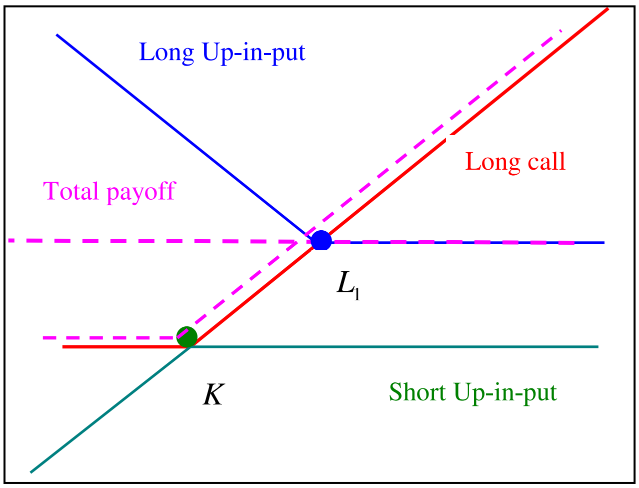
The dashed line in Figure 8 shows the possible payoffs that are possible. If L_1 is never breached this structure behaves like an ordinary vanilla call (the puts are not activated and do not exist) which means that if the expiry value of the underlying is below K, the payoff of the whole structure is zero. If L_1 is breached, $L_1 - K$ is guaranteed to the holder and, in essence, the vanilla call's strike level is increased to L_1. This is explained as follows:
the two up-and-in-puts are activated and are now vanilla put options. The short put and long call (both struck at K) together form a synthetic long position in a forward contract where $F = C(K) - P(K)$. The payoff for the ladder call now is just $LC = F + P(L_1)$. At expiry, if the underlying's price ends above the barrier, $P(L_1)$ is

out-the-money and the payoff is just the payoff for F. Actually $P(K)$ is also out-the-money and the total payoff is just that of the vanilla call i.e., $S_T - K$. If the price ends below the barrier, the total payoff is just the payoff for LO. The total payoff is given by

$$\text{payoff}\,(C(K) - P(K) + P(L_1)) = \max(0, S_T - K) - \max(0, K - S_T) + \max(0, L_1 - S_T).$$

This will always give a payoff of $L_1 - K$, for any $S_T \le L_1$ and $S_T - K$ if $S_T \ge L_1$.

Figure 8: The mechanics of a ladder call option



As a real example let's look at a 5 year SAPPI ladder call. SAPPI is currently trading at R52.80, the current volatility is 45%, interest is 15% and the dividend yield is 1.887%.
We assume the investor is not long the SAPPI stock and wants synthetic exposure to SAPPI as well as guaranteed pay-outs after the stock has gone up by certain amounts. The following lists a matrix of prices where the lock-in levels are 25%, 50%, 75% and 100% above the current price.

	Ladders	***Ladders***	***Ladders***	***Ladders***	***ATM Call***
	66.00	66.00	66.00	66.00	
		79.20	79.20	79.20	
			92.40	92.40	
				105.60	
Value	**29.76**	**31.73**	**33.44**	**34.92**	**27.46**
Value as %	56.35%	60.10%	63.33%	66.14%	52.00%

Here, the price is given for an increasing number of levels. In the last column we have given the value of an ATM vanilla call. The ladders are more expensive because of the lock-in feature at every rung.

C. Ladder Hedge

A ladder option is a synthetic structure where exposure to the market is achieved synthetically. We can however create an overlay structure from this if there is an underlying long position already in place. Using the formulae for the ladder options we do the following: replace the long call/put with a long put/call position which then leads to

$$RHC(K_0, L_i) = P(K_0) + \sum_{i=1}^{N} \left[UIP(L_i, L_i) - UIP(L_{i-1}, L_i)\right]$$

and

$$RHP(K_0, L_i) = C(K_0) + \sum_{i=1}^{N} \left[DIC(L_i, L_i) - DIC(L_{i-1}, L_i)\right]$$

This will actually give exactly the same hedges as that for the roll-up puts and roll-down calls with the extra enhancement that one does not need the last level as explained for the roll-ups and downs. With put-call-parity for barriers one can actually show that the equations for the roll options is equivalent to the ones just described. The behaviour is exactly the same as that for roll-up puts and roll-down calls.

As a real example let's look at a 5 year SAPPI roll-up put. SAPPI is currently trading at R52.80, the current volatility is 45%, interest is 15% and the dividend yield is 1.887%.
We assume the investor is long the SAPPI stock and wants to lock in the gains after the stock has gone up by certain amounts. The following lists a matrix of prices where the lock-in levels are 25%, 50%, 75% and 100% above the current price.

	Ladders	***Ladders***	***Ladders***	***Ladders***	***ATM Put***
	66.00	66.00	66.00	66.00	
		79.20	79.20	79.20	
			92.40	92.40	
				105.60	
Value	**7.92**	**9.89**	**11.60**	**13.08**	**5.62**
Value as %	15.00%	18.74%	21.97%	24.78%	10.65%

Here, the price is given for an increasing number of levels. In the last column we have given the value of an ATM vanilla call. The ladders are more expensive because of the lock-in feature at every rung. These prices are exactly the same as those for the roll-up puts.

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